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# The 2−Factoriality of the O'Grady Moduli Spaces

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**Abstract**

The aim of this work is to show that the moduli space $M_{10}$ introduced by O'Grady in [OG1] is a 2−factorial variety. Namely, $M_{10}$ is the moduli space of semistable sheaves with Mukai vector $v := (2, 0, -2) \in H^{ev}(X, \mathbb{Z})$ on a projective K3 surface $X$. As a corollary to our construction, we show that the Donaldson morphism gives a Hodge isometry between $v^{\perp}$ (sublattice of the Mukai lattice of $X$) and its image in $H^2(\widetilde{M}_{10}, \mathbb{Z})$, lattice with respect to the Beauville form of the 10−dimensional irreducible symplectic manifold $\widetilde{M}_{10}$, obtained as symplectic resolution of $M_{10}$. Similar results are shown for the moduli space $M_6$ introduced by O'Grady in [OG2].

## 1 Introduction

Moduli spaces of semistable sheaves on abelian or projective K3 surfaces are one of the main tools to produce examples of irreducible symplectic manifolds. If $M_v$ denotes the moduli space of semistable sheaves with Mukai vector $v$ on a projective K3 surface, it is a well-known result that if $v$ is primitive and the chosen polarization is $v$−generic, then $M_v$ is an irreducible symplectic manifold. Moreover, $M_v$ is deformation equivalent to an Hilbert scheme of points on some projective K3 surface. An analogous result shows that if the surface is abelian, from $M_v$ one can produce an irreducible symplectic manifold, which is deformation equivalent to a generalized Kummer variety on some abelian surface.

The choice of a non-primitive Mukai vector can give rise to new examples. Let $X$ be a projective K3 surface, and suppose there is an ample divisor $H$ on $X$ such that $H^2 = 2$ and $Pic(X) = \mathbb{Z} \cdot H$. Let us consider the moduli space $M_{10}$ of $H$−semistable sheaves on $X$ whose Mukai vector is $(2, 0, -2) \in H^{2*}(X, \mathbb{Z})$. The moduli space $M_{10}$ was introduced by O'Grady in [OG1], where he shows that $M_{10}$ admits a symplectic resolution $\widetilde{M}_{10}$, which is an irreducible symplectic manifold of dimension 10. In [OG2], O'Grady introduced a 6−dimensional moduli space $M_6$ of semistable sheaves on an abelian surface, showing that it admits a symplectic resolution $\widetilde{M}_6$,

which is an irreducible symplectic manifold of dimension 6. In both cases, the obtained manifold is not deformation equivalent to any other previously known example of irreducible symplectic manifold.

A natural question is if there are other moduli spaces of semistable sheaves admitting a symplectic resolution and giving rise to new irreducible symplectic manifolds. In [L-S] and [K-L-S], the authors answered to the question: in particular, in [L-S] it is shown that if $v = 2w$, where $w$ is a primitive Mukai vector such that $(w, w) = 2$, then $M_v$ admits a symplectic resolution, obtained as the blow-up of $M_v$ along its reduced singular locus. In [K-L-S] it is shown that if $v = mw$ for $m \in \mathbb{N}$ and $w$ a primitive Mukai vector, such that $m > 2$ or $m = 2$ and $(w, w) > 2$, then $M_v$ does not admit any symplectic resolution. In this case, $M_v$ is a locally factorial scheme.

The aim of this work is to describe singularities of $M_{10}$ and $M_6$. Namely, we show the following:

**Theorem 1.1.** *The moduli spaces $M_{10}$ and $M_6$ are $2-$factorial projective varieties.*

The main ingredient in the proof of Theorem 1.1 is the Le Potier morphism, which to certain classes in $K_{top}(X)$ associates a line bundle on a moduli space. In the case of $M_{10}$, using results of [R1] we show that the Weil divisor $B$ parameterizing non-locally free sheaves is not a Cartier divisor. Le Potier's construction allows us to show that $2B$ is Cartier. We will deduce the $2-$factoriality of $M_{10}$ from this. The same ideas are used in the proof of the $2-$factoriality of $M_6$, but here the problem is subtler: the exceptional divisor $\widetilde{\Sigma}$ of the symplectic resolution $\widetilde{M}_6$ is divisible by 2, while this is not the case for $M_{10}$. This property implies the existence of a Weil divisor $D$ on $M_6$ such that $2D = 0$. Using results of [R2], we show that $D$ is not a Cartier divisor and that $M_6$ is in fact $2-$factorial.

As a corollary to our construction, we show the following

**Theorem 1.2.** *Let $X$ be a projective K3 surface such that $Pic(X) = \mathbb{Z}\cdot H$ for some ample line bundle such that $H^2 = 2$, and let $v = (2, 0, -2) \in \widetilde{H}(X, \mathbb{Z})$. Let $v^\perp \subseteq \widetilde{H}(X, \mathbb{Z})$ be the orthogonal to $v$ with respect to the Mukai pairing. There is a Hodge injective morphism*

$$f : v^\perp \longrightarrow H^2(\widetilde{M}_{10}, \mathbb{Z})$$

*which gives an isometry between $v^\perp$ (lattice with respect to the Mukai form) and its image in $H^2(\widetilde{M}_{10}, \mathbb{Z})$ (lattice with respect to the Beauville form).*

An analogous result holds in the 6-dimensional example. This is the generalization of Theorem 0.1 in [Y] for moduli spaces $M_v$, with $v$ primitive.

The paper is organized as follows. Sections from 2 to 5 are devoted to the $10-$dimensional O'Grady's example: in section 2 we recall the construction

of $M_{10}$, and we show that it cannot be locally factorial. In section 3 and 4 we show that $M_{10}$ is $2-$factorial, and in section 5 we prove Theorem 1.2.

Sections from 6 to 9 are devoted to the $6-$dimensional O'Grady's example, following the same structure described for the previous example. The exposition of the two examples is presented as symmetric as possible, and the main proofs for the $6-$dimensional case are almost identical to those of the $10-$dimensional one. Anyway, subtle differencies are shown when necessary. Finally, in section 10 we present a brief appendix on constructions of flat families that we need all along the paper.

# 2 The local factoriality of $M_{10}$

In this section we recall the construction of the $10-$dimensional moduli space $M_{10}$ and its main properties, namely those contained in [R1]. We provide two construction of flat families of sheaves that we will use in sections 3 and 4, and we show that $M_{10}$ is not locally factorial.

## 2.1 Generalities on $M_{10}$

Let us recall the setting of [OG1]. Let $X$ be a projective K3 surface such that $Pic(X) = \mathbb{Z} \cdot H$, where $H$ is an ample line bundle with $H^2 = 2$. Let $M_{10}$ be the moduli space of $H-$semistable sheaves on $X$ with Mukai vector $(2, 0, -2)$. It is a $10-$dimensional projective variety whose regular locus is $M_{10}^s$, the open subset parameterizing stable sheaves. Let $\Sigma$ be the singular locus of $M_{10}$, which is a codimension 2 closed subset in $M_{10}$ (see [OG1]). As semistable locally free sheaves are stable in this setting (by Lemma 1.1.5 in [OG1]), the open subset $M_{10}^{lf}$ of $M_{10}$ parameterizing locally free sheaves is contained in $M_{10}^s$. Let $B$ be the closed subset of $M_{10}$ parameterizing non locally free sheaves: then $\Sigma \subseteq B$, and by [OG1], $B$ is an irreducible Weil divisor. The first result we need is:

**Theorem 2.1.** ***(O'Grady, Lehn-Sorger)*** *The moduli space $M_{10}$ admits a symplectic resolution $\pi : \widetilde{M}_{10} \longrightarrow M_{10}$, and $\widetilde{M}_{10}$ is a $10-$dimensional irreducible symplectic manifold with $b_2 \geq 24$. Moreover, $\widetilde{M}_{10}$ can be obtained as the blow-up of $M_{10}$ along $\Sigma$ with its reduced schematic structure.*

*Proof.* The proof is in [OG1]. In [L-S] it is proved that $\widetilde{M}_{10}$ can be obtained as the blow up of $M_{10}$ along its reduced singular locus. □

Let $\widetilde{\Sigma}$ be the exceptional divisor of $\pi$, and let $\widetilde{B}$ be the proper transform of $B$ under $\pi$. Let $M_{10}^{\mu-ss}$ be the Donaldson-Uhlenbeck compactification of the moduli space $M_{10}^{\mu}$ of $\mu-$stable sheaves, and let $\phi : M_{10} \longrightarrow M_{10}^{\mu-ss}$ be the canonical surjective morphism. As shown in [OG1], $M_{10}^{\mu-ss} = M_{10}^{lf} \coprod S^4(X)$.

Let $\delta$ be the fiber of $\pi$ over a generic point in the smooth locus of $\Sigma$, and let $\gamma'$ be the fiber of $\phi$ over a generic point of the smooth locus of $S^4(X) \subseteq M_{10}^{\mu-ss}$. Moreover, let $\gamma$ be the proper transform of $\gamma'$. Finally, let

$$\mu_D : H^2(X, \mathbb{Z}) \longrightarrow H^2(M_{10}^{\mu-ss}, \mathbb{Z})$$

be the Donaldson morphism (see [OG1], [F-M]).

**Theorem 2.2.** *(**Rapagnetta**). The second Betti number of $\widetilde{M}_{10}$ is 24. Moreover*

1. *The morphism $\widetilde{\mu} := \pi^* \circ \phi^* \circ \mu_D : H^2(X, \mathbb{Z}) \longrightarrow H^2(\widetilde{M}_{10}, \mathbb{Z})$ is injective.*
2. *We have the following equalities:*

$$c_1(\widetilde{\Sigma}) \cdot \delta = -2, \qquad c_1(\widetilde{B}) \cdot \delta = 1$$

$$c_1(\widetilde{\Sigma}) \cdot \gamma = 3, \qquad c_1(\widetilde{B}) \cdot \gamma = -2.$$

3. *The second integral cohomology of $\widetilde{M}_{10}$ is*

$$H^2(\widetilde{M}_{10}, \mathbb{Z}) = \widetilde{\mu}(H^2(X, \mathbb{Z})) \oplus \mathbb{Z} \cdot c_1(\widetilde{\Sigma}) \oplus \mathbb{Z} \cdot c_1(\widetilde{B}).$$

4. *Let $q$ be the Beauville form of $\widetilde{M}_{10}$. Then for every $\alpha, \beta \in H^2(X, \mathbb{Z})$ we have*

$$q(\widetilde{\mu}(\alpha), \widetilde{\mu}(\beta)) = \alpha \cdot \beta, \quad q(\widetilde{\mu}(\alpha), c_1(\widetilde{\Sigma})) = q(\widetilde{\mu}(\alpha), c_1(\widetilde{B})) = 0,$$

$$q(c_1(\widetilde{\Sigma}), c_1(\widetilde{\Sigma})) = -6, \quad q(c_1(\widetilde{\Sigma}), c_1(\widetilde{B})) = 3,$$
$$q(c_1(\widetilde{B}), c_1(\widetilde{\Sigma})) = 3, \quad q(c_1(\widetilde{B}), c_1(\widetilde{B})) = -2.$$

*Proof.* The proof is contained in [R1], Theorems 1.1, 3.1 and 4.3. □

## 2.2 Flat families

In this subsection we present two examples of flat families of sheaves we will use in the following. We refer to section 10 for the general construction.

*Example* 2.1. Let $X$ be a projective K3 surface with $Pic(X) = \mathbb{Z} \cdot H$, where $H$ is an ample line bundle such that $H^2 = 2$. Fix three different points $x_1, x_2, x_3 \in X$, and consider

$$i : X \longrightarrow S^4(X), \quad i(x) := x + x_1 + x_2 + x_3,$$

which is a closed immersion. Let $T := i(X) \simeq X$, and consider a surjective morphism $\varphi : \mathscr{O}_X^2 \longrightarrow \mathbb{C}_{x_1} \oplus \mathbb{C}_{x_2} \oplus \mathbb{C}_{x_3}$ as in Proposition 3.0.5 in [OG1].

Let $\mathscr{K} := \ker(\varphi)$, which is a rank 2 sheaf such that $det(\mathscr{K}) = \mathscr{O}_X$ and $c_2(\mathscr{K}) = 3$.

Let $\Delta \subseteq T \times X$ be the diagonal (up to the isomorphism between $T$ and $X$). By Corollary 2, Chapter II.5 in [M] (see Lemma 5.5 below), the sheaf $p_{T*}\mathscr{H}om(p_X^*\mathscr{K}, \mathscr{O}_\Delta)$ is a rank 2 vector bundle, and for every $x \in T$ the canonical morphism

$$(p_{T*}\mathscr{H}om(p_X^*\mathscr{K}, \mathscr{O}_\Delta))_x \longrightarrow Hom(\mathscr{K}, \mathbb{C}_x)$$

is an isomorphism. Let $Y := \mathbb{P}(p_{T*}\mathscr{H}om(p_X^*\mathscr{K}, \mathscr{O}_\Delta))$ and $p : Y \longrightarrow T$ be the canonical projection. We have a canonical morphism (see Section 10)

$$\widetilde{f} : q_X^*\mathscr{K} \otimes q_Y^*\mathscr{T} \longrightarrow (p \times id_X)^*\mathscr{O}_\Delta,$$

where $q_X$ and $q_Y$ are the natural projections of $Y \times X$ to $X$ and $Y$ respectively, and $\mathscr{T}$ is the tautological line bundle on $Y$. Consider $\mathscr{H} := \ker(\widetilde{f})$.

**Lemma 2.3.** *Let $\mathscr{E}$ be a sheaf defining a point in $B$ and whose singular locus is given by $x, x_1, x_2, x_3$. Then $\mathscr{E}$ defines a point $[f_\mathscr{E}] \in Y$, and the restriction $\mathscr{H}_{[f_\mathscr{E}]} := \mathscr{H}_{|q_Y^{-1}([f_\mathscr{E}])}$ is isomorphic to $\mathscr{E}$. Moreover, the morphism $\widetilde{f}$ is surjective and $\mathscr{H}$ is a $Y-$flat family.*

*Proof.* The sheaf $\mathscr{E}$ is the kernel of a surjective (hence non-zero) morphism $f_\mathscr{E} : \mathscr{K} \longrightarrow \mathbb{C}_x$ (see section 3.1 in [OG1]), defining a point $t = [f_\mathscr{E}] \in p^{-1}(x)$ since $p^{-1}(x) \simeq \mathbb{P}(Hom(\mathscr{K}, \mathbb{C}_x))$. By definition of $\widetilde{f}$, we have $\widetilde{f}_t = f_\mathscr{E}$. The morphism $\widetilde{f}$ is surjective: indeed, $\mathrm{coker}(\widetilde{f})$ is trivial if and only if it is trivial on the fibers of $q_Y$. If $t \in Y$, then $t$ corresponds to a surjective morphism $f_\mathscr{E}$, so that $\mathrm{coker}(\widetilde{f})_t = \mathrm{coker}(f_\mathscr{E}) = 0$, and we are done.

Since $\widetilde{f}$ is surjective, the family $\mathscr{H}$ is $Y-$flat. Now, since $q_X^*\mathscr{K} \otimes q_Y^*\mathscr{T}$ and $(p \times id_X)^*\mathscr{O}_\Delta$ are $Y-$flat, for every $t \in Y$ the canonical morphism

$$\mathscr{H}_t \longrightarrow (q_X^*\mathscr{K} \otimes q_Y^*\mathscr{T})_t \simeq \mathscr{K}$$

is injective, so that $\mathscr{H}_t = \ker(\widetilde{f}_t)$. As $t \in Y$ corresponds to a surjective morphism $f_\mathscr{E} : \mathscr{K} \longrightarrow \mathbb{C}_x$, where $x = p(t)$, whose kernel is $\mathscr{E}$, and $\widetilde{f}_t = f_\mathscr{E}$, we are done. $\square$

*Example* 2.2. In the same setting of the previous example, let $x \in X$ be different from $x_1, x_2, x_3$, and let $T := \{x\}$. Moreover, let $i$ be the inclusion of $T$ in $X$. Let $Y := \mathbb{P}(p_{T*}\mathscr{H}om(p_X^*\mathscr{K}, i_*\mathbb{C}_x)) \simeq \mathbb{P}^1$, By the general construction, we have $\widetilde{f} : q_X^*\mathscr{K} \longrightarrow (p \times id_X)^* i_*\mathbb{C}_x \otimes q_Y^*\mathscr{O}(1)$, where $p : Y \longrightarrow T$ is the canonical morphism. In particular, notice that $(p \times id_X)^* i_*\mathbb{C}_x \otimes q_Y^*\mathscr{O}(1) = j_*\mathscr{O}(1)$, where $j : \mathbb{P}^1 \times T \longrightarrow \mathbb{P}^1 \times X$ is the inclusion. In conclusion, we have $\widetilde{f} : q_X^*\mathscr{K} \longrightarrow j_*\mathscr{O}(1)$. Finally, let $\mathscr{H} := \ker(\widetilde{f})$.

**Lemma 2.4.** *Let $\mathscr{E}$ be a sheaf defining a point in $B$ whose singular locus is given by $x, x_1, x_2, x_3$, and let $[f_{\mathscr{E}}]$ be the point of $Y$ defined by $\mathscr{E}$. Then $\mathscr{H}_{[f_{\mathscr{E}}]} \simeq \mathscr{E}$, and $\widetilde{f}$ is a surjective morphism. Moreover, the family $\mathscr{H}$ is $Y-$flat.*

*Proof.* The proof works as the one of Lemma 2.3. □

## 2.3 The moduli space $M_{10}$ is not locally factorial

A first application of Theorem 2.2 is the following:

**Proposition 2.5.** *If $n \in \mathbb{Z}$ is such that $nB$ is a Cartier divisor, then $n$ is even. In particular, $M_{10}$ is not locally factorial.*

*Proof.* Let $n \in \mathbb{Z}$ be such that $nB$ is Cartier. Then $\pi^*(nB) = n\widetilde{B} + m\widetilde{\Sigma}$ for some $m \in \mathbb{Z}$, since $\widetilde{B}$ is the proper transform of $B$. By the projection formula we have $c_1(\pi^*(nB)) \cdot \delta = 0$, as $\delta$ is contracted by $\pi$. By point 2 of Theorem 2.2, we get

$$0 = c_1(\pi^*(nB)) \cdot \delta = nc_1(\widetilde{B}) \cdot \delta + mc_1(\widetilde{\Sigma}) \cdot \delta = n - 2m,$$

so that $n$ is even. Finally, $M_{10}$ is not locally factorial: if it was, then $B$ would be a Cartier divisor, which is clearly not the case by the previous part of the proof. □

*Remark* 2.1. Theorem 2.2 implies even that $Pic(M_{10})$ is free. Indeed, let $L \in Pic(M_{10})$ be torsion of period $t \in \mathbb{N}$, and let $\widetilde{L}$ be its proper transform under $\pi$. Then $\pi^*(L) = \widetilde{L} + m\widetilde{\Sigma}$ for some $m \in \mathbb{Z}$, and $t(\widetilde{L} + m\widetilde{\Sigma}) = 0$. As $\widetilde{M}_{10}$ is simply connected, by point 3 of Theorem 2.2 we see that $Pic(\widetilde{M}_{10})$ is free: in conclusion $\widetilde{L} = -m\widetilde{\Sigma}$, so that $L = 0$. The same argument shows that $\pi^* : Pic(M_{10}) \longrightarrow Pic(\widetilde{M}_{10})$ is injective.

Moreover, $c_1 : Pic(M_{10}) \longrightarrow H^2(M_{10}, \mathbb{Z})$ is injective: if $L, L' \in Pic(M_{10})$ are such that $c_1(L) = c_1(L')$, then $c_1(\pi^*(L)) = c_1(\pi^*(L'))$, so that $\pi^*(L) = \pi^*(L')$. As $\pi^*$ is injective on $Pic(M_{10})$, this implies $L \simeq L'$.

To conclude this section, we show the following:

**Lemma 2.6.** *If $n \in \mathbb{Z}$ is such that $nB$ is a Cartier divisor, then*

$$c_1(nB) \cdot \gamma' = -\frac{n}{2} \in \mathbb{Z}.$$

*Proof.* The fact that $-n/2$ is an integer follows from Proposition 2.5, as $nB$ is a Cartier divisor. By definition of $\gamma$ and $\gamma'$, there is $l \in \mathbb{Q}$ such that

$\pi^*(\gamma') = \gamma + l\delta$. By point 2 in Theorem 2.2 and the projection formula we have

$$3 = c_1(\widetilde{\Sigma}) \cdot \gamma = c_1(\widetilde{\Sigma}) \cdot \pi^*(\gamma') - l(c_1(\widetilde{\Sigma}) \cdot \delta) = 2l,$$

so that $\pi^*(\gamma') = \gamma + \frac{3}{2}\delta$. Now, suppose that $n \in \mathbb{Z}$ is such that $nB$ is a Cartier divisor. By the projection formula $c_1(nB) \cdot \gamma' = nc_1(\widetilde{B}) \cdot \pi^*(\gamma')$, so that

$$c_1(nB) \cdot \gamma' = nc_1(\widetilde{B}) \cdot \gamma + \frac{3n}{2} c_1(\widetilde{B}) \cdot \delta = -2n + \frac{3n}{2} = -\frac{n}{2},$$

by point 2 of Theorem 2.2, and we are done. $\square$

# 3 Line bundles on $M_{10}$

In this section we study properties of line bundles on $M_{10}$. The main ingredients are Le Potier's construction of line bundles on moduli spaces of sheaves on algebraic surfaces, and the construction of flat families we presented in section 2.2.

## 3.1 Le Potier's construction

We recall Le Potier's construction (see [LeP] or [H-L], Chapter 8). Let $S$ be a Noetherian scheme, and let $\mathscr{F}$ be an $S-$flat family on $S \times X$. We can define a morphism

$$\widetilde{\lambda} : K_{top}(X) \longrightarrow Pic(S), \quad \widetilde{\lambda}(\alpha) := det(p_{R!}(p_X^* \alpha \cdot [\mathscr{F}])).$$

We apply this construction when $S$ is the open subset $R$ of a Grothendieck Quot-scheme whose quotient is $M_{10}$, and $\mathscr{F}$ is a universal family on $R \times X$. Let $e := [\mathscr{E}] \in K_{top}(X)$ be the class of a sheaf $\mathscr{E}$ parameterized by $M_{10}$, $h := [H] \in K_{top}(X)$ and let

$$\xi : K_{top}(X) \times K_{top}(X) \longrightarrow \mathbb{Z}, \quad \xi(\alpha, \beta) := \chi(\alpha \cdot \beta).$$

By Theorem 8.1.5 in [H-L], $\widetilde{\lambda}(\alpha) \in Pic(R)$ descends to a line bundle $\lambda(\alpha) \in Pic(M_{10})$ if $\alpha \in e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$ (orthogonality with respect to $\xi$), so that there is a group morphism

$$\lambda : e^{\perp} \cap \{1, h, h^2\}^{\perp\perp} \longrightarrow Pic(M_{10}).$$

The first result we need is:

**Lemma 3.1.** *Let $\alpha \in K_{top}(X)$. Then $\alpha \in e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$ if and only if $c_1(\alpha) \in Pic(X)$ and $ch_2(\alpha) = 0$.*

*Proof.* By the Hirzebruch-Riemann-Roch Theorem, a class $\beta \in K_{top}(X)$ is in $\{1, h, h^2\}^{\perp}$ if and only if $v(\beta) = (0, b, 0)$, where $b \in H^2(X, \mathbb{Z})$ is such that $b \cdot c_1(H) = 0$. Then $\beta \in H^2(X, \mathbb{Z}) \cap (H^{2,0}(X) \oplus H^{0,2}(X))$, as $Pic(X) = \mathbb{Z} \cdot H$.

Now, let $\alpha \in K_{top}(X)$. Then $\alpha \in \{1, h, h^2\}^{\perp\perp}$ if and only if $\chi(\alpha \cdot \beta) = 0$ for every $\beta \in \{1, h, h^2\}^{\perp}$. By the previous part, we get $c_1(\alpha) \cdot b = 0$ for every $b \in H^2(X, \mathbb{Z}) \cap (H^{2,0}(X) \oplus H^{0,2}(X))$. Then $c_1(\alpha)$ has to be the first Chern class of a line bundle on $X$. Finally, by the Hirzebruch-Riemann-Roch Theorem, we have $\alpha \in e^{\perp}$ if and only if $ch_2(\alpha) = 0$, as $ch(e) = (2, 0, -4)$. □

Using this lemma, we are able to prove the following:

**Proposition 3.2.** *Let $p \in X$ be any point, and let*

$$u_1 : Pic(X) \longrightarrow e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}, \quad u_1(L) := [\mathscr{O}_X - L] + \frac{c_1^2(L)}{2}[\mathbb{C}_p],$$

$$u_2 : \mathbb{Z} \longrightarrow e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}, \quad u_2(n) := n[\mathscr{O}_X].$$

*Then $u := u_1 + u_2$ is a group isomorphism.*

*Proof.* Let $L \in Pic(X)$. The Mukai vector of $u_1(L)$ is $(0, -c_1(L), 0)$, so that $u_1(L) \in e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$ by Lemma 3.1. Moreover, for every $L_1, L_2 \in Pic(X)$ we have $v(u_1(L_1 \otimes L_2)) = v(u_1(L_1) + u_1(L_2))$, where $v : K_{top}(X) \longrightarrow H^{2*}(X, \mathbb{Z})$ is the morphism sending a class in $K_{top}(X)$ to its Mukai vector. As $v$ is a group isomorphism (see [K]), then $u_1$ is a group morphism.

Let $n \in \mathbb{Z}$. The Mukai vector of $u_2(n)$ is $v(u_2(n)) = (n, 0, n)$, so that $u_2(n) \in e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$ by Lemma 3.1, and $u_2$ is clearly a group morphism.

We need to show that $u$ is an isomorphism. For the injectivity, let $(L, n), (M, m) \in Pic(X) \oplus \mathbb{Z}$ be such that $u(L, n) = u(M, m)$. Their Mukai vectors are then equal: as these are, respectively, $(n, -c_1(L), n)$ and $(m, -c_1(M), m)$, this implies $n = m$ and $c_1(L) = c_1(M)$. As $X$ is a K3 surface, this implies $L \simeq M$, and injectivity is shown. For the surjectivity, let $\alpha \in e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$: by Lemma 3.1, we have $v(\alpha) = (r, c_1(L), r)$ for some $r \in \mathbb{Z}$ and $L \in Pic(X)$. Then $v(\alpha) = v(u(L^{-1}, r))$, and $\alpha = u(L^{-1}, r)$. □

**Proposition 3.3.** *We have the following intersecion properties.*

1. *For every $L \in Pic(X)$, we have $c_1(\lambda(u_1(L))) \cdot \gamma' = 0$.*
2. *For every $n \in \mathbb{Z}$ we have $c_1(\lambda(u_2(n))) \cdot \gamma' = -n$.*

*Proof.* We begin with the first item. As $Pic(X) = \mathbb{Z} \cdot H$, we need to verify the statement only for $H$. By Proposition 8.2.3 in [H-L] there is $m \in \mathbb{N}$ such that $\lambda(u(H))^{\otimes m}$ is generated by its global sections, and the canonical map

$$\phi : M_{10} \longrightarrow \mathbb{P}(H^0(M_{10}, \lambda(u(H))^{\otimes m})^*)$$

has $M_{10}^{\mu-ss}$ as image. In particular, $\phi^*\mathscr{O}(1) = \lambda(u(H))^{\otimes m}$, so that

$$mc_1(\lambda(u(H))) \cdot \gamma' = c_1(\lambda(u(H))^{\otimes m}) \cdot \gamma' = c_1(\phi^*\mathscr{O}(1)) \cdot \gamma' = 0,$$

as $\gamma'$ is contracted by $\phi$. Finally $c_1(\lambda(u(H))) \cdot \gamma' = 0$, and we are done.

For the second item, we need to verify the statement only for $n = 1$. Notice that $c_1(\lambda(u_2(1))) \cdot \gamma' = c_1(\lambda([\mathscr{O}_X])_{|\gamma'})$. Using the family $\mathscr{H}$ defined in Example 2.2, we have

$$c_1(\lambda([\mathscr{O}_X])_{|\gamma'}) = c_1(q_{Y!}(q_X^*[\mathscr{O}_X] \cdot [\mathscr{H}])).$$

By the Grothendieck-Riemann-Roch Theorem, as the fibers of $q_Y$ are of dimension 2 we have

$$c_1(q_{Y!}(q_X^*[\mathscr{O}_X] \cdot [\mathscr{H}])) = q_{Y*}[q_X^*(ch(\mathscr{O}_X)td(X)^{-1}) \cdot ch(\mathscr{H})]_3 =$$

$$= -2q_X^*[y] \cdot ch_1(\mathscr{H}) + ch_3(\mathscr{H}),$$

where $[y]$ is the class of a point in $X$. By the Grothendieck-Riemann-Roch Theorem and by definition of $\mathscr{H}$ we have $ch_1(\mathscr{H}) = [j_*ch(\mathscr{O}_{\mathbb{P}^1}(1))]_1$ and

$$ch_3(\mathscr{H}) = [j_*ch(\mathscr{O}_{\mathbb{P}^1}(1))]_3 - 2q_X^*[y] \cdot [j_*ch(\mathscr{O}_{\mathbb{P}^1}(1))]_1.$$

In conclusion, $ch_1(\mathscr{H}) = 0$ and $ch_3(\mathscr{H}) = -q_Y^*[p]$, where $[p]$ is the class of a point in $Y$. Finally, we get

$$c_1(\lambda([\mathscr{O}_X])) \cdot \gamma' = q_{Y*}(-q_Y^*[p]) = -1,$$

and we are done. □

## 3.2 Donaldson's and Le Potier's morphisms

The aim of this section is to prove that the morphism $\lambda \circ u$ is injective. The main result we need is the following:

**Proposition 3.4.** *Let $L \in Pic(X)$. Then $c_1(\lambda(u_1(L))) = \phi^*\mu_D(c_1(L))$.*

*Proof.* The proof is done in two steps: first we show that these two classes are equal when restricted to a well-chosen subvariety; then we show that the equality on this restriction implies the equality everywhere.

*Step 1.* Here, we refer to Example 2.1 for the notations. Consider the inclusion $i : T \simeq X \longrightarrow M_{10}^{\mu-ss}$ described in Example 2.1, and consider the morphism $j : Y \longrightarrow M_{10}$ induced by the family $\mathscr{H}$. By Lemma 2.3, $j$ is injective and its image is $\phi^{-1}(T)$. For every $L \in Pic(X)$ we have

$$j^*\phi^*(\mu_D(c_1(L))) = p^*i^*(\mu_D(c_1(L))).$$

By Proposition 6.5 in [F-M] we have $i^*(\mu_D(c_1(L))) = c_1(L) \in NS(X)$ (up to the isomorphism between $X$ and $T$), and we need to show that $j^*c_1(\lambda(u_1(L))) = p^*(c_1(L))$. By Theorem 8.1.5 in [H-L] and Lemma 2.3

$$j^*\lambda(u_1(L)) = det(q_{Y!}(q_X^*u_1(L)\cdot[\mathscr{H}])),$$

so that by the Grothendieck-Riemann-Roch Theorem we get

$$c_1(j^*\lambda(u_1(L))) = q_{Y*}[q_X^*(ch(u_1(L))td(X)^{-1})\cdot ch(\mathscr{H})]_3 =$$

$$= -q_X^*(c_1(L))\cdot ch_2(\mathscr{H}).$$

By Lemma 2.3 we finally have

$$q_{Y*}(-q_X^*(c_1(L))\cdot ch_2(\mathscr{H})) = q_{Y*}(q_X^*(c_1(L))\cdot(p\times id_X)^*[\Delta]) = p^*(c_1(L)).$$

*Step 2.* Let $L \in Pic(X)$ and $\beta := \phi^*\mu_D(c_1(L)) - c_1(\lambda(u_1(L))) \in H^2(M_{10},\mathbb{Z})$. We need to show that $\beta = 0$.

By Step 1, $j^*\beta = 0$. Moreover, $\beta\cdot\gamma' = 0$: indeed, $\phi^*\mu_D(c_1(L))\cdot\gamma' = 0$ as $\gamma'$ is contracted by $\phi$, and $c_1(\lambda(u_1(L)))\cdot\gamma' = 0$ by point 1 of Proposition 3.3. Now, by point 3 of Theorem 2.2, there are $\alpha\in H^2(X,\mathbb{Z})$ and $n,m\in\mathbb{Z}$ such that $\pi^*\beta = \widetilde{\mu}(\alpha) + nc_1(\widetilde{\Sigma}) + mc_1(\widetilde{B})$. By point 2 of Theorem 2.2, we get

$$0 = \pi^*\beta\cdot\delta = \widetilde{\mu}(\alpha)\cdot\delta + nc_1(\widetilde{\Sigma})\cdot\delta + mc_1(\widetilde{B})\cdot\delta = m - 2n$$

as $\delta$ is contracted by $\pi$, and

$$0 = \pi^*\beta\cdot\gamma = \widetilde{\mu}(\alpha)\cdot\gamma + nc_1(\widetilde{\Sigma})\cdot\gamma + mc_1(\widetilde{B})\cdot\gamma = 3n - 2m$$

since $\pi^*\beta\cdot\gamma = \beta\cdot\gamma' = 0$. In conclusion, $n = m = 0$ and $\pi^*\beta = \widetilde{\mu}(\alpha)$. This implies $\beta = \phi^*\mu_D(\alpha)$: indeed, $\beta$ and $\phi^*\mu_D(\alpha)$ are in $NS(M_{10})$, and $\pi^*$ is injective on $NS(M_{10})$ by Remark 2.1. Restricting to $Y$ we then get

$$0 = j^*\beta = j^*\phi^*\mu_D(\alpha) = p^*(\alpha),$$

the last equality coming from Proposition 6.5 in [F-M]. To conclude, simply note that $p^* : NS(T) \longrightarrow NS(Y)$ is injective as $Y$ is a $\mathbb{P}^1-$bundle on $T$, so that $\alpha = 0$, and we are done. ☐

**Corollary 3.5.** *The morphism* $\lambda\circ u : Pic(X)\oplus\mathbb{Z} \longrightarrow Pic(M_{10})$ *is injective. Moreover, we have* $Pic(\widetilde{M}_{10}) = \pi^*\circ\lambda\circ u(Pic(X))\oplus\mathbb{Z}[\widetilde{\Sigma}]\oplus\mathbb{Z}[\widetilde{B}]$.

*Proof.* For the injectivity of $\lambda\circ u$, let $L, M \in Pic(X)$ and $n, m\in\mathbb{Z}$ be such that $\lambda(u(L,n)) = \lambda(u(M,m))$. By Proposition 3.3 we have

$$-n = c_1(\lambda(u(L,n)))\cdot\gamma' = c_1(\lambda(u(M,m)))\cdot\gamma' = -m,$$

so that $m = n$ and $\lambda(u_1(L)) = \lambda(u_1(M))$. In particular, $c_1(\lambda(u_1(L))) = c_1(\lambda(u_1(M)))$, so that $\phi^*\mu_D(c_1(L)) = \phi^*\mu_D(c_1(M))$ by Proposition 3.4. Now, by point 1 of Theorem 2.2, the morphism $\phi^* \circ \mu_D$ is injective, so that we finally get $c_1(L) = c_1(M)$, implying $L \simeq M$ as $X$ is a K3 surface.

By Remark 2.1 the morphism $\pi^* : Pic(M_{10}) \longrightarrow Pic(\widetilde{M}_{10})$ is injective. Moreover, as $\lambda \circ u$ is injective, the morphism $\lambda \circ u_1$ is, so that

$$\pi^* \circ \lambda \circ u_1 : Pic(X) \longrightarrow Pic(\widetilde{M}_{10})$$

is injective. To conclude, let $L \in Pic(\widetilde{M}_{10})$. By point 3 of Theorem 2.2, there are $\alpha \in H^2(X, \mathbb{Z})$ and $n, m \in \mathbb{Z}$ such that $c_1(L) = \widetilde{\mu}(\alpha) + nc_1(\widetilde{\Sigma}) + mc_1(\widetilde{B})$. In particular $\widetilde{\mu}(\alpha) \in NS(\widetilde{M}_{10})$, so that $\phi^*\mu_D(\alpha) \in NS(M_{10})$. By Proposition 6.5 in [F-M] $j^*\phi^*\mu_D(\alpha) = p^*(\alpha)$, so that $p^*(\alpha) \in NS(Y)$. But this implies $\alpha \in NS(X)$, and we are done. □

# 4 The 2−factoriality of $M_{10}$

Using the results of the previous section, we are finally able to show the 2−factoriality of $M_{10}$.

**Proposition 4.1.** *Let $A^1(M_{10})$ be the group of Weil divisors of $M_{10}$ modulo linear equivalence. Then $A^1(M_{10}) = \lambda(u_1(Pic(X))) \oplus \mathbb{Z}[B]$.*

*Proof.* Notice that $A^1(M_{10}) \simeq Pic(\pi^{-1}(M_{10}^s)$. Indeed, $\pi$ is an isomorphism on $M_{10}^s$, so that $Pic(\pi^{-1}(M_{10}^s)) \simeq Pic(M_{10}^s)$, and $Pic(M_{10}^s) = A^1(M_{10}^s)$, as $M_{10}^s$ is smooth. Since $\Sigma = M_{10} \setminus M_{10}^s$ has codimension 2 in $M_{10}$, we have $A^1(M_{10}^s) = A^1(M_{10})$. Now, let us consider the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\sigma} Pic(\widetilde{M}_{10}) \xrightarrow{\rho} Pic(\pi^{-1}(M_{10}^s)) \longrightarrow 0,$$

where $\sigma(1) := [\widetilde{\Sigma}]$ and $\rho$ is the restriction morphism. We claim that it is exact: $\sigma$ is clearly injective and $\rho$ is surjective (see [H], Proposition 6.5). Moreover, $\widetilde{\Sigma} \in \ker(\rho)$. Let $L \in \ker(\rho)$: we need to show that $L$ is a multiple of $\widetilde{\Sigma}$. By Corollary 3.5 there are $M \in Pic(X)$ and $n, m \in \mathbb{Z}$ such that

$$L = \pi^*(\lambda(u_1(M))) + n\widetilde{B} + m\widetilde{\Sigma}.$$

As $\rho(\widetilde{\Sigma}) = 0$, we get $\rho(L) = \lambda(u_1(M)) + nB$ as Weil divisors on $M_{10}$, so that $nB = -\lambda(u_1(M))$. In particular, $nB$ is a Cartier divisor: by Lemma 2.6 and point 1 of Proposition 3.3, we have

$$-\frac{n}{2} = nB \cdot \gamma' = \lambda(u_1(M)) \cdot \gamma' = 0,$$

so that $n = 0$ and $M = \mathscr{O}_X$ (by Proposition 3.4). By Corollary 3.5 we are done. □

**Corollary 4.2.** *The only Weil divisors on $M_{10}$ that are possibly not Cartier are the multiples of $B$.*

*Proof.* As $\lambda(u_1(Pic(X))) \subseteq Pic(M_{10})$, this is an immediate corollary of Proposition 4.1. □

The final result of this section is the following, which is one of the main results of the paper:

**Theorem 4.3.** *There is $L \in Pic(X)$ such that $2B = \lambda(u(L, 1)) \in Pic(M_{10})$. In particular, the moduli space $M_{10}$ is $2-$factorial.*

*Proof.* As $B$ is not a Cartier divisor by Proposition 2.5, by Corollary 4.2 the $2-$factoriality of $M_{10}$ follows once we show $2B \in Pic(M_{10})$. By Proposition 4.1, there are $n \in \mathbb{Z}$ and $M \in Pic(X)$ such that $\lambda(u_2(1)) = \lambda(u_1(M)) + nB$. In particular $nB \in Pic(M_{10})$, so that by Proposition 3.3 and Lemma 2.6

$$-1 = c_1(\lambda(u_2(1))) \cdot \gamma' = c_1(\lambda(u_1(M))) \cdot \gamma' + c_1(nB) \cdot \gamma' = -\frac{n}{2}.$$

In conclusion $n = 2$ and $2B = \lambda(u(M^{-1}, 1))$. □

# 5 The Beauville form of $\widetilde{M}_{10}$

The aim of this section is to show that the line bundle $L$ in the statement of Theorem 4.3 is trivial. As a consequence of this, we prove Theorem 1.2.

## 5.1 Properties of the Weil divisor $B$

In this section we show some properties of the sheaves parameterized by $M_{10}$. The main result is that the Weil divisor $B$ is characterized in cohomological terms. We begin with two lemmas.

**Lemma 5.1.** *Let $E$ be a rank 2 locally free sheaf with trivial determinant. Then $E \simeq E^*$.*

*Proof.* By hypothesis on $E$, the canonical morphism $E \otimes E \longrightarrow E \wedge E$ is a perfect pairing. □

**Lemma 5.2.** *Let $\mathscr{E}$ be a sheaf defining a point in $M_{10}$. Then $H^0(X, \mathscr{E}) = 0$ and $h^1(X, \mathscr{E}) = h^2(X, \mathscr{E})$.*

*Proof.* By the Hirzebruch-Riemann-Roch Theorem, the Hilbert polynomial of $\mathscr{E}$ is $P(\mathscr{E},n) = 2n^2$, since $H^2 = 2$. In particular, $\chi(\mathscr{E}) = 0$, so that $h^1(X,\mathscr{E}) = h^0(X,\mathscr{E}) + h^2(X,\mathscr{E})$. We show that $h^0(X,\mathscr{E}) = 0$. Recall that the reduced Hilbert polynomial of a sheaf $\mathscr{F}$ of dimension 2 on a surface is $p(\mathscr{F},n) := P(\mathscr{F},n)/rk(\mathscr{F})$: then $p(\mathscr{E},n) = n^2$, and $p(\mathscr{O}_X,n) = n^2+2$. By Proposition 1.2.7 in [H-L], we have $Hom(\mathscr{O}_X,\mathscr{E}) = 0$, and we are done. $\square$

As a consequence, we have the following:

**Proposition 5.3.** *Let $\mathscr{E}$ be a semistable sheaf with Mukai vector $(2,0,-2)$.*

1. *If $\mathscr{E}$ is locally free, then $H^i(X,\mathscr{E}) = 0$ for $i = 0,1,2$.*
2. *If $\mathscr{E}$ is not locally free, then $h^1(X,\mathscr{E}) = h^2(X,\mathscr{E}) \neq 0$.*

*Proof.* By Lemma 5.2 we have $H^0(X,\mathscr{E}) = 0$ and $h^1(X,\mathscr{E}) = h^2(X,\mathscr{E})$. If $\mathscr{E}$ is locally free, by Serre's duality $h^2(X,\mathscr{E}^*) = 0$. Then $h^2(X,\mathscr{E}) = 0$ by Lemma 5.1, and the first item is shown. If $\mathscr{E}$ is not locally free, we have two cases.

*Case 1*: $[\mathscr{E}] \in B \cap M^s_{10}$. Then $\mathscr{E}^{**} = \mathscr{O}_X \oplus \mathscr{O}_X$, and we have a short exact sequence

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{O}_X \oplus \mathscr{O}_X \longrightarrow \mathscr{G} \longrightarrow 0$$

since $\mathscr{E}$ is torsion free, where $\mathscr{G}$ is supported on a finite number of points. Thus $h^2(X,\mathscr{E}) = 2$, and we are done.

*Case 2*: $\mathscr{E}$ is strictly semistable. By Lemma 1.1.5 in [OG1], $\mathscr{E}$ fits into an exact sequence

$$0 \longrightarrow \mathscr{I}_Z \longrightarrow \mathscr{E} \longrightarrow \mathscr{I}_W \longrightarrow 0 \tag{1}$$

for some $Z,W \in Hilb^2(X)$. Since $H^0(X,\mathscr{I}_W) = 0$ and $h^i(X,\mathscr{I}_Z) = 1$ for $i = 1,2$, the long exact sequence induced by (1) implies $h^2(X,\mathscr{E}) \neq 0$. $\square$

Let $\mathscr{F}$ be a universal family on $R \times X$, and consider the universal quotient module

$$0 \longrightarrow \mathscr{G} \longrightarrow p_X^*\mathscr{H} \xrightarrow{\rho} \mathscr{F} \longrightarrow 0, \tag{2}$$

where $p_X$ is the projection on $X$ and $\mathscr{H} := H^0(X,\mathscr{E}(NH)) \otimes \mathscr{O}_X(-NH)$ for $N \in \mathbb{Z}$ sufficiently big, where $\mathscr{E}$ is any sheaf parameterized by $M_{10}$. In particular, $\mathscr{H}$ is locally free and $H^0(X,\mathscr{H}) = H^1(X,\mathscr{H}) = 0$. Notice that any $s \in R$ corresponds to an exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow \mathscr{H} \xrightarrow{f_{\mathscr{E}}} \mathscr{E} \longrightarrow 0. \tag{3}$$

As $\mathscr{F}$ and $p_X^*\mathscr{H}$ are $R-$flat, the sheaf $\mathscr{G}$ is $R-$flat. For any $s \in R$ let $\mathscr{G}_s$ (resp. $(p_X^*\mathscr{H})_s$, $\mathscr{F}_s$) denote the restriction of $\mathscr{G}$ (resp. $p_X^*\mathscr{H}$, $\mathscr{F}$) to the fiber of the projection $p_R : R \times X \longrightarrow R$ over the point $s$. Then

$$\mathscr{G}_s \simeq \ker((p_X^*\mathscr{H})_s \longrightarrow \mathscr{F}_s) = \ker(f_{\mathscr{E}}) = \mathscr{K}.$$

**Proposition 5.4.** *We have the following properties:*

1. *For every $i \in \mathbb{Z}$ the sheaves $\mathbb{R}^i p_{R*}\mathscr{G}$ and $\mathbb{R}^i p_{R*}(p_X^*\mathscr{H})$ are locally free of rank $h^i(X, \mathscr{H})$.*

2. *For every $s \in R$ and $i \in \mathbb{Z}$, the canonical morphism*

$$(\mathbb{R}^i p_{R*}\mathscr{F})_s \longrightarrow H^i(p_R^{-1}(s), \mathscr{F}_s) \simeq H^i(X, \mathscr{E})$$

*is an isomorphism, where $\mathscr{E}$ is a sheaf corresponding to the point $s \in R$.*

*Proof.* The main ingredient is the following lemma:

**Lemma 5.5.** *Let $f : T \longrightarrow S$ be a proper morphism of Noetherian schemes, and suppose $S$ to be reduced. Let $\mathscr{U} \in Coh(T)$ be an $S-$flat family of sheaves, and let $i \in \mathbb{Z}$. The function sending any $s \in S$ to $h^i(T_s, \mathscr{U}_s)$ is constant if and only if $\mathbb{R}^i f_*\mathscr{U}$ is locally free and the canonical morphism*

$$(\mathbb{R}^i f_*\mathscr{U})_s \longrightarrow H^i(T_s, \mathscr{U}_s)$$

*is an isomorphism. If this is verified, then $(\mathbb{R}^{i-1} f_*\mathscr{U})_s \longrightarrow H^{i-1}(T_s, \mathscr{U}_s)$ is an isomorphism for every $s \in S$.*

*Proof.* See [M], Chapter II.5, Corollary 2. $\square$

We only need to show the proposition for $i = 0, 1, 2$. For every $s \in R$ we have $(p_X^*\mathscr{H})_s \simeq \mathscr{H}$, so that $H^i(p_R^{-1}(s), (p_X^*\mathscr{H})_s) \simeq H^i(X, \mathscr{H})$, and the function sending $s \in R$ to $h^i(p_R^{-1}(s), (p_X^*\mathscr{H})_s)$ is constant. By Lemma 5.5, the sheaf $\mathbb{R}^i p_{R*}(p_X^*\mathscr{H})$ is locally free of rank $h^i(X, \mathscr{H})$. In particular, as $h^0(X, \mathscr{H}) = h^1(X, \mathscr{H}) = 0$, then $\mathbb{R}^0 p_{R*}(p_X^*\mathscr{H}) = \mathbb{R}^1 p_{R*}(p_X^*\mathscr{H}) = 0$.

The next step is to study $\mathbb{R}^i p_{R*}\mathscr{G}$. Applying $\mathbb{R}p_{R*}$ to the exact sequence (2), by the first part of the proposition we get $\mathbb{R}^0 p_{R*}\mathscr{G} = 0$ and $\mathbb{R}^1 p_{R*}\mathscr{G} \simeq \mathbb{R}^0 p_{R*}\mathscr{F}$. We show that this last sheaf is trivial. Let $\mathscr{E}$ be a sheaf parameterized by $M_{10}$, and consider a corresponding point $s \in R$. Then $\mathscr{F}_s \simeq \mathscr{E}$, and the map sending $s$ to $H^0(X, \mathscr{F}_s)$ is constant and trivial by Lemma 5.2. The canonical morphism $(\mathbb{R}^0 p_{R*}(\mathscr{F}))_s \longrightarrow H^0(X, \mathscr{F}_s) = 0$ is then an isomorphism by Lemma 5.5, so that $\mathbb{R}^0 p_{R*}(\mathscr{F}) = 0$. It remains to show that $\mathbb{R}^2 p_{R*}\mathscr{G}$ is a vector bundle of rank $h^2(X, \mathscr{H})$: consider $s \in R$

and its associated exact sequence (3). The long exact sequence induced by this and Lemma 5.2 imply $h^2(X,\mathscr{G}_s) = h^2(X,\mathscr{H})$, so that $\mathbb{R}^2 p_{R*}\mathscr{G}$ is a vector bundle of rank $h^2(X,\mathscr{H})$; for every $s \in R$ the canonical morphism $(\mathbb{R}^2 p_{R*}\mathscr{G})_s \longrightarrow H^2(X,\mathscr{G}_s) \simeq H^2(X,\mathscr{K})$ is an isomorphism by Lemma 5.5.

Finally, we study $\mathbb{R}^i p_{R*}\mathscr{F}$ for $i = 1, 2$. As $\mathbb{R}^3 p_{R*}\mathscr{F} = 0$, by Lemma 5.5 the canonical morphism $(\mathbb{R}^2 p_{R*}\mathscr{F})_s \longrightarrow H^2(X,\mathscr{E})$ is an isomorphism. Let $\xi : \mathbb{R}^1 p_{R*}\mathscr{F} \longrightarrow \mathbb{R}^2 p_{R*}\mathscr{G}$ be the morphism induced by the exact sequence (2). Since $\mathbb{R}^1 p_{R*}(p_X^*\mathscr{H}) = 0$ by the first part of the proof, $\xi$ is injective. In particular, for any $s \in R$ the morphism $\xi_s$ is injective, so that

$$(\mathbb{R}^1 p_{R*}(\mathscr{F}))_s \simeq \ker((\mathbb{R}^2 p_{R*}(\mathscr{G}))_s \xrightarrow{\delta} (\mathbb{R}^2 p_{R*}(p_X^*\mathscr{H}))_s).$$

The morphism $\delta$ is simply the morphism $H^2(X,\mathscr{K}) \longrightarrow H^2(X,\mathscr{H})$ induced by the exact sequence (3), by the previous part of the proof. Since $H^1(X,\mathscr{H}) = 0$, we have $\ker(\delta) \simeq H^1(X,\mathscr{E})$, and we are done. $\square$

We are finally able to prove the following

**Proposition 5.6.** *We have* $2B = \lambda(u_2(1))$.

*Proof.* By definition $\widetilde{\lambda}(u_2(1)) = det(\mathbb{R}p_{R*}(\mathscr{F}))$. By Theorem 4.3, the line bundle $\widetilde{\lambda}(u_2(1))$ descends to $2B + \lambda(u_1(L))$ for some $L \in Pic(X)$. Applying $\mathbb{R}p_{R*}$ to the exact sequence (2), by point 1 of Proposition 5.4 we get the exact sequence

$$0 \longrightarrow \mathbb{R}^1 p_{R*}(\mathscr{F}) \longrightarrow \mathbb{R}^2 p_{R*}(\mathscr{G}) \xrightarrow{\beta} \mathbb{R}^2 p_{R*}(p_X^*\mathscr{H}) \longrightarrow \mathbb{R}^2 p_{R*}(\mathscr{F}) \longrightarrow 0.$$

As $\mathbb{R}^0 p_{R*}\mathscr{F} = 0$ by point 2 of Proposition 5.4, we get $det(\mathbb{R}p_{R*}\mathscr{F}) \simeq det(\mathbb{R}^2 p_{R*}(p_X^*\mathscr{H})) \otimes det(\mathbb{R}^2 p_{R*}\mathscr{G})^{-1}$. Then $det(\beta)$ gives a section $s$ of the line bundle $det(\mathbb{R}p_{R*}(\mathscr{F}))$, whose zero locus is given by the set where $det(\beta)$ is not an isomorphism. By Propositions 5.3 and 5.4 this locus is exactly $p^{-1}(B)$, and we are done. $\square$

## 5.2 Description of $H^2(\widetilde{M}_{10},\mathbb{Z})$

Yoshioka (see Theorem 0.1 in [Y]) showed the following: if $S$ is any projective K3 surface, $v \in H^{2*}(S,\mathbb{Z})$ is a primitive Mukai vector with $(v,v) > 0$ and $H$ is a $v-$generic polarization, the moduli space $M_v$ of $H-$semistable sheaves on $S$ with Mukai vector $v$ is an irreducible symplectic variety, and there is an isometry of Hodge structures $v^\perp \longrightarrow H^2(M_v,\mathbb{Z})$, where $v^\perp$ is a sublattice of the Mukai lattice of $S$ and $H^2(M_v,\mathbb{Z})$ is a lattice with respect to the Beauville form. In this section, we show an analogue of this in the case of $\widetilde{M}_{10}$. Here, $X$ is a projective K3 surface with Picard group spanned by an ample line bundle $H$ such that $H^2 = 2$, and $v = (2,0,-2) \in H^{2*}(X,\mathbb{Z})$.

**Lemma 5.7.** *We have $v^{\perp} \simeq H^2(X,\mathbb{Z}) \oplus \mathbb{Z}$.*

*Proof.* By the Hirzebruch-Riemann-Roch Theorem, $w \in H^{2*}(X,\mathbb{Z})$ is orthogonal to $v$ if and only if $w = (r,c,r)$ for $r \in \mathbb{Z}$ and $c \in H^2(X,\mathbb{Z})$. $\square$

We have the Hodge morphism $\widetilde{\mu} : H^2(X,\mathbb{Z}) \longrightarrow H^2(\widetilde{M}_{10},\mathbb{Z})$ respecting the lattice structures, and the morphism

$$c_1 \circ \pi^* \circ \lambda \circ u : Pic(X) \oplus \mathbb{Z} \longrightarrow H^2(\widetilde{M}_{10},\mathbb{Z}).$$

By Proposition 3.4, these two morphisms agree on $Pic(X)$. Let

$$f : v^{\perp} \longrightarrow H^2(\widetilde{M}_{10},\mathbb{Z}), \qquad f(r,c,r) := \widetilde{\mu}(c) + c_1(\pi^*(\lambda(u_2(r)))).$$

**Theorem 5.8.** *The morphism $f$ is a Hodge isometry between $v^{\perp}$, viewed as a sublattice of the Mukai lattice $\widetilde{H}(X,\mathbb{Z})$, and its image in $H^2(\widetilde{M}_{10},\mathbb{Z})$, being a lattice with respect to the Beauville form $q$.*

*Proof.* The morphism $f$ is an injective morphism of Hodge structures by point 1 of Theorem 2.2. By Proposition 5.6, $\lambda(u_2(r)) = 2rB$, so that

$$\pi^*\lambda(u_2(r)) = 2r\widetilde{B} + m\widetilde{\Sigma}$$

for some $m \in \mathbb{Z}$. Intersecting with $\delta$, by point 2 of Theorem 2.2 we get $m = r$. In conclusion, we have $f(r,c,r) = \widetilde{\mu}(c) + 2rc_1(\widetilde{B}) + rc_1(\widetilde{\Sigma})$. By point 4 of Theorem 2.2 and by definition of the Mukai pairing, it is then an easy calculation to show that $f$ is an isometry. $\square$

# 6 The local factoriality of $M_6$

From now on, we deal with the $6-$dimensional O'Grady's example $M_6$, and we show that it is $2-$factorial. In this section we recall the construction of $M_6$ and of $\widetilde{M}_6$, and we resume the basic properties we need for the proof of the $2-$factoriality. Moreover, we show that $M_6$ is not locally factorial.

## 6.1 Generalities on $M_6$

In the following, let $C$ be a smooth projective curve of degree 2, and let $J := Pic^0(C)$ be its jacobian surface. Suppose there is an ample line bundle $H$ on $J$ such that $NS(J) = \mathbb{Z} \cdot c_1(H)$ and $c_1^2(H) = 2$. Finally, let $\widehat{J} := Pic^0(J)$ be the abelian surface dual to $J$.

Let $v := (2,0,-2) \in \widetilde{H}(J,\mathbb{Z})$, and let $M_v$ be the moduli space of $H-$semistable sheaves on $J$ whose Mukai vector is $v$. The regular locus

of $M_v$ is the open subset $M_v^s$ parameterizing stable sheaves. Let $\Sigma_v$ be the singular locus of $M_v$, which is a closed subset of codimension 2 in $M_v$ (see [OG2]). Since in this setting any semistable locally free sheaf is stable (see Lemma 2.1.2 in [OG2]), the open subset $M_v^{lf}$ of $M_v$ parmaterizing locally free sheaves is contained in $M_v^s$. Let $\overline{B}_v$ be the closed subset of $M_v$ parameterizing non-locally free sheaves. In particular, $\Sigma_v \subseteq \overline{B}_v$. Finally, let

$$a_v : M_v \longrightarrow J \times \widehat{J}, \qquad a_v([\mathscr{E}]) := \Big(\sum c_2(\mathscr{E}), det(\mathscr{E})\Big),$$

and let $M_6 := a_v^{-1}(0, \mathscr{O}_J)$.

**Theorem 6.1.** ***(O'Grady, Lehn-Sorger)****. The moduli space $M_v$ admits a symplectic resolution $\pi_v : \widetilde{M}_v \longrightarrow M_v$, which is obtained as the blow-up of $M_v$ along $\Sigma_v$ with reduced schematic structure. Let $\widetilde{M}_6 := \pi_v^{-1}(M_6)$. Then $\widetilde{M}_6$ is an irreducible symplectic variety of dimension 6 and second Betti number 8.*

*Proof.* The proof is in [OG2]. In [L-S] it is show that $\widetilde{M}_v$ can be obtained as the blow up of $M_v$ along its reduced singular locus. □

Let $\widetilde{\Sigma}_v$ be the exceptional divisor of $\pi_v$, and let $\widetilde{B}_v$ be the proper transform of $\overline{B}_v$ under $\pi_v$. Let $\pi := \pi_{v|\widetilde{M}_6}$, and let $\Sigma := \Sigma_v \cap M_6$, the singular locus of $M_6$. In particular, $\pi$ is the blow up of $M_6$ along $\Sigma$ with its reduced structure. Finally, let $\overline{B} := \overline{B}_v \cap M_6$, $\widetilde{\Sigma} := \pi^{-1}(\Sigma)$ (the exceptional divisor of $\pi$) and $\widetilde{B} := \widetilde{B}_v \cap \widetilde{M}_6$ (the proper transform of $\overline{B}$ under $\pi$). As shown in section 5.1 in [OG2], $\widetilde{B}$ is an irreducible Weil divisor on $\widetilde{M}_6$. Let $M_6^{\mu-ss}$ be the Donaldson-Uhlenbeck compactification of the moduli space $M_6^{\mu}$ of $\mu-$stable sheaves, and let $\phi : M_6 \longrightarrow M_6^{\mu-ss}$ be the canonical surjective morphism. Let $\delta$ be the fiber of $\pi$ over a generic point in the smooth locus of $\Sigma$, and let $\gamma$ be as in section 5.1 of [OG2]. Finally, let

$$\mu_D : H^2(J, \mathbb{Z}) \longrightarrow H^2(M_6^{\mu-ss}, \mathbb{Z})$$

be the Donaldson morphism.

**Theorem 6.2.** ***(Rapagnetta)****. Let $\widetilde{\mu} := \pi^* \circ \phi^* \circ \mu_D$.*

1. *The morphism $\widetilde{\mu} : H^2(J, \mathbb{Z}) \longrightarrow H^2(\widetilde{M}_6, \mathbb{Z})$ is injective.*
2. *There is a line bundle $A \in Pic(\widetilde{M}_6)$ such that $c_1(\widetilde{\Sigma}) = 2c_1(A)$.*
3. *We have the following equalities:*

$$c_1(A) \cdot \delta = -1, \qquad c_1(\widetilde{B}) \cdot \delta = 1,$$
$$c_1(A) \cdot \gamma = 1, \qquad c_1(\widetilde{B}) \cdot \gamma = -2.$$

*4. The second integral cohomology of $\widetilde{M}_6$ is*

$$H^2(\widetilde{M}_6, \mathbb{Z}) = \widetilde{\mu}(H^2(J, \mathbb{Z})) \oplus \mathbb{Z} \cdot c_1(A) \oplus \mathbb{Z} \cdot c_1(\widetilde{B}).$$

*5. Let $q$ be the Beauville form of $\widetilde{M}_6$. Then for every $\alpha, \beta \in H^2(J, \mathbb{Z})$ we have*

$$q(\widetilde{\mu}(\alpha), \widetilde{\mu}(\beta)) = \alpha \cdot \beta, \quad q(\widetilde{\mu}(\alpha), c_1(\widetilde{B})) = q(\widetilde{\mu}(\alpha), c_1(A)) = 0,$$

$$q(c_1(A), c_1(A)) = -2, \qquad q(c_1(A), c_1(\widetilde{B})) = 2,$$
$$q(c_1(\widetilde{B}), c_1(A)) = 2, \qquad q(c_1(\widetilde{B}), c_1(\widetilde{B})) = -4.$$

*Proof.* Item 1 is [OG2], Proposition 7.3.3. The proof of the other points is contained in [R2], Theorems 3.3.1, 3.4.1 and 3.5.1. □

## 6.2 Flat families

In this subsection we present two examples of flat families of sheaves we will use in the following. As in section 2.2, we refer to section 10 for the general construction.

*Example* 6.1. Let $E$ be a rank 2 vector bundle on $J$ with trivial first and second Chern classes and such that $hom(E, E) = 2$. Moreover, let $J[2]$ be the set of 2−torsion points in $J$, and let $y \in J \setminus J[2]$. Fix a surjective morphism $\varphi : E \longrightarrow \mathbb{C}_y$ and let $\mathscr{K} := \ker(\varphi)$: by Lemma 4.3.3 in [OG2], any sheaf defining a point in $\widetilde{B}_v$ is the kernel of a surjective morphism from $\mathscr{K}$ to $\mathbb{C}_x$ for some point $x \in J$. Let $p_1, p_2 : J \times J \longrightarrow J$ be the two projections. As in Example 2.1, the sheaf $p_{1*}\mathscr{H}om(p_2^*\mathscr{K}, \mathscr{O}_\Delta)$ is a vector bundle of rank 2, and for any $x \in J$ the canonical morphism

$$(p_{1*}\mathscr{H}om(p_2^*\mathscr{K}, \mathscr{O}_\Delta))_x \longrightarrow Hom(\mathscr{K}, \mathbb{C}_x)$$

is an isomorphism (see Lemma 5.5). Let $Y := \mathbb{P}(p_{1*}\mathscr{H}om(p_2^*\mathscr{K}, \mathscr{O}_\Delta)) \xrightarrow{p} J$. There is a tautological morphism (see Section 10)

$$\widetilde{f} : q_J^*\mathscr{K} \otimes q_Y^*\mathscr{T} \longrightarrow (p \times id_J)^*\mathscr{O}_\Delta,$$

whose kernel is denoted $\mathscr{H}$.

**Lemma 6.3.** *Let $\mathscr{E}$ be a sheaf defining a point in $\widetilde{B}_v$ whose bidual is $E$ and whose singular locus is given by $x, y \in J$. Let $f_\mathscr{E} : \mathscr{K} \longrightarrow \mathbb{C}_x$ be the surjective morphism whose kernel is $\mathscr{E}$. Then $f_\mathscr{E}$ defines a point $[f_\mathscr{E}] \in Y$, and $\mathscr{H}_{[f_\mathscr{E}]} \simeq \mathscr{E}$. Moreover, $\mathscr{H}$ is a $Y$−flat family and $\widetilde{f}$ is surjective.*

*Proof.* The proof is the same as the one of Lemma 2.3. □

*Example* 6.2. Let $E$ be as in the previous example, with the further property that $det(E) \simeq \mathscr{O}_J$. Let $x \in J$ and $\varphi : E \longrightarrow \mathbb{C}_{-x}$, a surjective morphism whose kernel is denoted $\mathscr{K}$. Let $Y := \mathbb{P}(p_{x*}\mathscr{H}om(p_J^*\mathscr{K}, i_*\mathbb{C}_x)) \xrightarrow{p} \{x\}$, where $p_J : \{x\} \times J \longrightarrow J$ and $p_x : \{x\} \times J \longrightarrow \{x\}$ are the two projections, and $i : \{x\} \longrightarrow J$ is the closed immersion. Then, $Y \simeq \mathbb{P}^1$, and its points correspond to surjective morphisms from $\mathscr{K}$ to $\mathbb{C}_x$. As before, we get a tautological morphism $\widetilde{f} : q_J^*\mathscr{K} \longrightarrow j_*\mathscr{O}_{\mathbb{P}^1}(1)$, where $j : \mathbb{P}^1 \times \{x\} \longrightarrow \mathbb{P}^1 \times J$ is the immersion. Let $\mathscr{H} := \ker(\widetilde{f})$.

**Lemma 6.4.** *Let $\mathscr{E}$ be a sheaf defining a point in $\widetilde{B}$ whose bidual is $E$ and whose singular locus is given by $x, -x \in J$. Let $f_{\mathscr{E}} : \mathscr{K} \longrightarrow \mathbb{C}_x$ be the surjective morphism whose kernel is $\mathscr{E}$. Then $f_{\mathscr{E}}$ defines a point $[f_{\mathscr{E}}] \in Y$, and $\mathscr{H}_{[f_{\mathscr{E}}]} \simeq \mathscr{E}$. Moreover, $\mathscr{H}$ is a $Y-$flat family and $\widetilde{f}$ is surjective.*

*Proof.* Again, the proof is the same as the one of Lemma 2.3, using the Claim in section 5.1 of [OG2]. □

### 6.3 The moduli space $M_6$ is not locally factorial

A first application of Theorem 6.2 is the following:

**Lemma 6.5.** *There is a non-trivial irreducible Weil divisor $D \in A^1(M_6)$ such that $2D = 0$. If $\widetilde{D}$ is the proper transform of $D$ by $\pi$, then there is $m \in \mathbb{Z}$ such that $A = \widetilde{D} + m\widetilde{\Sigma}$ in the group $Div(\widetilde{M}_6)$ of Weil divisors of $\widetilde{M}_6$.*

*Proof.* As in the proof of Proposition 4.1, we have $A^1(M_6) \simeq Pic(\pi^{-1}(M_6^s))$. The restriction of $A$ to $\pi^{-1}(M_6^s)$ defines then an irreducible Weil divisor $D \in A^1(M_6)$. By point 2 of Theorem 6.2 we have

$$2D = 2A_{|\pi^{-1}(M_6^s)} = \widetilde{\Sigma}_{|\pi^{-1}(M_6^s)} = 0.$$

Now, the Weil divisor $\widetilde{\Sigma}$ is a prime divisor, so it is a generator for the group $Div(\widetilde{M}_6)$. Since $A$ is a line bundle on $\widetilde{M}_6$, it defines an element in $Div(\widetilde{M}_6)$, so that there are $m, m_1, ..., m_n \in \mathbb{Z}$ and prime divisors $D_1, ..., D_n$ such that

$$A = m\widetilde{\Sigma} + \sum_{i=1}^{n} m_i D_i.$$

As $A_{|\pi^{-1}(M_6^s)} = \sum_{i=1}^n m_i D_{i|\pi^{-1}(M_6^s)}$, we have $\sum_{i=1}^n m_i D_i = \widetilde{D}$, and we are done. It remains to show that $D$ is not trivial: if $D = 0$, then $\widetilde{D} = 0$, so that $A = m\widetilde{\Sigma} = 2mA$ (by point 2 of Theorem 6.2). Then $c_1(A)$ is torsion in $H^2(\widetilde{M}_6, \mathbb{Z})$, which is not possible by point 4 of Theorem 6.2. □

**Proposition 6.6.** *The Weil divisor $D$ is not Cartier, and $M_6$ is not locally factorial.*

*Proof.* If $D$ was a Cartier divisor, then $\pi^*(D) = \widetilde{D} + kA$, for some $k \in \mathbb{Z}$. By Lemma 6.5 we then get $\pi^*(D) = (1-2m+k)A$. The integer $1-2m+k$ is odd: indeed, if there was $n \in \mathbb{Z}$ such that $2n = 1-2m+k$, then $\pi^*(D) = n\widetilde{\Sigma}$ and we would have

$$D = \pi^*(D)_{|\pi^{-1}(M_6^s)} = n\widetilde{\Sigma}_{|\pi^{-1}(M_6^s)} = 0,$$

which is not possible since $D$ is non-trivial. By point 3 of Theorem 6.2 and the fact that $\delta$ is contracted by $\pi$, one gets

$$0 = c_1(\pi^*(D)) \cdot \delta = (1 - 2m + k)c_1(A) \cdot \delta = 2m - k - 1.$$

As $2m - k - 1 \neq 0$, we get a contradiction, and $D$ is not a Cartier divisor. Finally, this clearly implies that $M_6$ cannot be locally factorial. □

*Remark* 6.1. As a consequence of this, $Pic(M_6)$ is free. Indeed, let $L \in Pic(M_6)$ be torsion of period $t$, and let $\widetilde{L}$ be its proper transform under $\pi$. Then $\pi^*(L) = \widetilde{L} + kA$ for some $k \in \mathbb{Z}$, and $t(\widetilde{L} + kA) = 0$. As $Pic(\widetilde{M}_6)$ has no torsion by point 4 of Theorem 6.2, we get $\widetilde{L} = -kA$, and

$$L = \widetilde{L}_{|\pi^{-1}(M_6^s)} = -kA_{|\pi^{-1}(M_6^s)} = -kD.$$

As $L \in Pic(M_6)$, we get $kD \in Pic(M_6)$, so that $k$ has to be even by Proposition 6.6 and Lemma 6.5. In conclusion $L = 0$, and we are done.

The same proof even shows that $\pi^* : Pic(M_6) \longrightarrow Pic(\widetilde{M}_6)$ is injective. As in Remark 2.1, from this one can deduce that the morphism $c_1 : Pic(M_6) \longrightarrow H^2(M_6, \mathbb{Z})$ is injective.

# 7 Line bundles on $M_6$

In this section we calculate the Picard groups of $\widetilde{M}_6$ and of $M_6$ following the same argument as in sections 3 and 4.

## 7.1 Le Potier's construction

Let $e := [\mathscr{E}] \in K_{top}(J)$ be the class of a sheaf $\mathscr{E}$ parameterized by $M_6$, and let $h := [H] \in K_{top}(J)$.

**Lemma 7.1.** *Let $\alpha \in K_{top}(J)$. Then $\alpha \in e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}$ if and only if $c_1(\alpha) \in c_1(H)^{\perp\perp}$ and $ch_2(\alpha) = rk(\alpha)\eta_J \in H^4(J, \mathbb{Z})$, where $\eta_J$ is the fundamental class of $J$.*

*Proof.* The proof works as the one of Lemma 3.1. $\square$

Using this lemma, we are able to prove the following:

**Proposition 7.2.** *Let $p \in J$ be any point, and let*

$$u_1 : Pic(J) \longrightarrow e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}, \quad u_1(L) := [\mathscr{O}_J - L] + \frac{c_1^2(L)}{2}[\mathbb{C}_p],$$

$$u_2 : \mathbb{Z} \longrightarrow e^{\perp} \cap \{1, h, h^2\}^{\perp\perp}, \quad u_2(n) := n[\mathscr{O}_J] + n[\mathbb{C}_p].$$

*The morphism $u := u_1 + u_2$ is a group isomorphism.*

*Proof.* The proof works as the one of Proposition 3.2. $\square$

**Proposition 7.3.** *Let $i : M_6 \longrightarrow M_v$ be the inclusion. There is a group morphism $\widetilde{\lambda}_6 := i^* \circ \lambda \circ u : Pic(J) \oplus \mathbb{Z} \longrightarrow Pic(M_6)$, where $\lambda$ is the Le Potier morphism. In particular, this induces a group morphism*

$$\lambda_6 : NS(J) \oplus \mathbb{Z} \longrightarrow Pic(M_6)$$

*such that for any $L \in Pic(J)$, $n \in \mathbb{Z}$ we have $\lambda_6(c_1(L), n) = \widetilde{\lambda}_6(L, n)$.*

*Proof.* The existence of the maps $\lambda \circ u$ and $\widetilde{\lambda}_6$ is implied by Lemma 7.1 and Theorem 8.1.5 in [H-L]. The fact that $\lambda \circ u$, and hence $\widetilde{\lambda}_6$, is a group morphism is as in the proof of Proposition 3.2. As $v : K_{top}(J) \longrightarrow H^{2*}(J, \mathbb{Z})$ is an isomorphism, if $c_1(L_1) = c_1(L_2)$, then $u(L_1) = u(L_2)$, and we are done. But this implies the existence of the morphism $\lambda_6$ defined on $NS(J) \oplus \mathbb{Z}$, and we are done. $\square$

In the following, let $\widetilde{\lambda}_{6,1} := i^* \circ \lambda \circ u_1$ and $\widetilde{\lambda}_{6,2} := i^* \circ \lambda \circ u_2$, so that $\widetilde{\lambda}_6 = \widetilde{\lambda}_{6,1} + \widetilde{\lambda}_{6,2}$. Then $\widetilde{\lambda}_{6,1}$ induces a morphism $\lambda_{6,1} : NS(J) \longrightarrow Pic(M_6)$, such that for every $L \in Pic(J)$ we have $\lambda_{6,1}(c_1(L)) = \widetilde{\lambda}_{6,1}(L)$. Then we have $\lambda_6 = \lambda_{6,1} + \widetilde{\lambda}_{6,2}$.

**Lemma 7.4.** *We have the following intersection properties.*

1. *Let $L \in Pic(J)$. Then $c_1(\pi^* \widetilde{\lambda}_{6,1}(L))) \cdot \gamma = c_1(\pi^* \widetilde{\lambda}_{6,1}(L)))) \cdot \delta = 0$.*
2. *Let $n \in \mathbb{Z}$. Then $c_1(\pi^* \widetilde{\lambda}_{6,2}(n)) \cdot \gamma = -n$.*

*Proof.* We begin with the first item. The equality $c_1(\pi^* \widetilde{\lambda}_6(L)) \cdot \delta = 0$ is trivial, as $\delta$ is contracted by $\pi$. Notice that

$$c_1(\pi^* \widetilde{\lambda}_{6,1}(L)) \cdot \gamma = c_1(\pi^* \widetilde{\lambda}_{6,1}(L)_{|\gamma}) = c_1(\lambda_{\mathscr{H}}(u_1(L)))$$

by Theorem 8.1.5 in [H-L] and Lemma 6.4, where $\lambda_{\mathscr{H}}$ is the Le Potier's morphism defined using the flat family $\mathscr{H}$ of Example 6.2. By the Grothendieck-Riemann-Roch Theorem, we have

$$c_1(\lambda_{\mathscr{H}}(u_1(L))) = q_{Y*}[q_J^*(ch(u_1(L))td(J)) \cdot ch(\mathscr{H}))]_3 \in H^2(\mathbb{P}^1, \mathbb{Z}) =$$

$$= -q_{Y*}(q_J^*(c_1(L)) \cdot ch_2(\mathscr{H})) = 0,$$

and we are done. For the second item, the proof is the same as the one of point 2 of Proposition 3.3, using Example 6.2. □

### 7.2 Donaldson's and Le Potier's morphisms

The main result we need is the following:

**Proposition 7.5.** *For any $L \in Pic(J)$ we have $c_1(\pi^* \widetilde{\lambda}_{6,1}(L)) = \widetilde{\mu}(c_1(L))$.*

*Proof.* The proof of this proposition is almost the same as the one of Proposition 3.4. Let $L \in Pic(J)$ and let $Y$ and $\mathscr{H}$ be as in Example 6.1. Using the same argument as in Step 1 of the proof of Proposition 3.4, we get

$$c_1(\pi^* \lambda(u_1(L)))_{|Y} = \widetilde{\mu}(c_1(L))_{|Y}. \tag{4}$$

Now, let $Y_6 := Y \cap \widetilde{M}_6$, and let $\beta := c_1(\pi^* \widetilde{\lambda}_{6,1}(L)) - \widetilde{\mu}(c_1(L)) \in H^2(\widetilde{M}_6, \mathbb{Z})$. By equation (4), we have $\beta_{|Y_6} = 0$, and by point 1 of Lemma 7.4 and the definition of $\widetilde{\mu}$ we have $\beta \cdot \gamma = \beta \cdot \delta = 0$. Following Step 2 of the proof of Proposition 3.4, these two properties imply $\beta = 0$, and we are done. □

**Corollary 7.6.** *The morphism $\lambda_6 : NS(J) \oplus \mathbb{Z} \longrightarrow Pic(M_6)$ is injective. Moreover, we have $Pic(\widetilde{M}_6) = \pi^* \lambda_{6,1}(NS(J)) \oplus \mathbb{Z} \cdot [A] \oplus \mathbb{Z} \cdot [\widetilde{B}]$.*

*Proof.* The proof works as that of Corollary 3.5, using Proposition 7.5. □

# 8 The 2−factoriality of $M_6$

We are now able to show the 2−factoriality of $M_6$. We need to add a remark on $\overline{B}$: the proper transform of $\overline{B}$ is an irreducible Weil divisor in $\widetilde{M}_6$, so that $\overline{B} = \Sigma \cup B$ for some irreducible Weil divisor $B$ of $M_6$ whose proper transform is $\widetilde{B}$.

**Proposition 8.1.** *We have $A^1(M_6) = \lambda_6(NS(J)) \oplus \mathbb{Z} \cdot [B] \oplus \mathbb{Z}/2\mathbb{Z} \cdot [D]$.*

*Proof.* The proof is similar to the one of Proposition 4.1, and we need to show that the following sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\sigma} Pic(\widetilde{M}_6) \xrightarrow{\rho} Pic(\pi^{-1}(M_6^s)) \longrightarrow 0$$

is exact, where $\sigma(1) := \widetilde{\Sigma}$ and $\rho$ is the restriction morphism. The only thing to prove is that if $L \in \ker(\rho)$, then it is a multiple of $\widetilde{\Sigma}$. By Corollary 7.6, there are $M \in Pic(J)$ and $n, m \in \mathbb{Z}$ such that $L = \pi^*(\lambda_{6,1}(c_1(M))) + n\widetilde{B} + mA$. By Lemma 6.5, we have $\rho(L) = \lambda_{6,1}(c_1(M)) + nB + mD \in A^1(M_6)$. As $\rho(L) = 0$, then $\rho(2L) = 0$, so that $2nB = \lambda_{6,1}(2c_1(M))$, as $2mD = 0$ by Lemma 6.5. In particular, their proper transforms are equal, getting $2n\widetilde{B} = \pi^*(\lambda_{6,1}(2c_1(M)))$, so that

$$-4n = 2nc_1(\widetilde{B}) \cdot \gamma = c_1(\pi^*\lambda_{6,1}(2c_1(M)) \cdot \gamma = 0,$$

by point 3 of Theorem 6.2 and point 1 of Lemma 7.4. In conclusion, $n = 0$ and $\lambda_{6,1}(2c_1(M)) = 0$. By Corollary 7.6 then $c_1(M) = 0$. In conclusion $L = mA$ for some $m \in \mathbb{Z}$, so that

$$0 = \rho(L) = \rho(mA) = mD.$$

By Lemma 6.5, then, $m$ is even and $L$ is a multiple of $\widetilde{\Sigma}$. $\square$

Here is the main result of this section:

**Theorem 8.2.** *There is a line bundle $L \in Pic(J)$ and $t \in \mathbb{Z}/2\mathbb{Z}$ such that $B + tD = \lambda_6(c_1(L), 1)$. In particular, $M_6$ is $2-$factorial.*

*Proof.* By Lemma 7.1 and Proposition 8.1 there are $M \in Pic(J)$, $n \in \mathbb{Z}$ and $t \in \mathbb{Z}/2\mathbb{Z}$ such that $\widetilde{\lambda}_{6,2}(1) = \lambda_{6,1}(c_1(M)) + nB + tD \in A^1(M_6)$. In particular $nB + tD \in Pic(M_6)$: we need to show that $n = 1$. Taking the pull-back of $nB + tD$ to $\widetilde{M}_6$ there is $m \in \mathbb{Z}$ such that

$$n\widetilde{B} + mA = \pi^*(nB + tD) = \pi^*(\lambda_6(-c_1(M), 1)).$$

By point 3 of Theorem 6.2 we get

$$0 = \pi^*(nB + tD) \cdot \delta = n\widetilde{B} \cdot \delta + mA \cdot \delta = n - m,$$

as $\delta$ is contracted by $\pi$, and

$$-2n + m = n\widetilde{B} \cdot \gamma + mA \cdot \gamma = \pi^*(\lambda_6(-c_1(M), 1)) \cdot \gamma = -1$$

by Lemma 7.4. In conclusion, $n = 1$ and we are done. It remains to show that $M_6$ is $2-$factorial: since $B + tD$ is a Cartier divisor, we have

$$\lambda_6(NS(J)) \oplus \mathbb{Z}[B + tD] \subseteq Pic(M_6).$$

We have then two possibilities: the first one is $t = 0$, so that $B$ is Cartier. In this case, the only Weil divisor which is not Cartier is $D$, and we are done. The second case is $t = 1$, so that $B + D$ is Cartier. As $2D = 0$, we then get $2B \in Pic(M_6)$, and we are done. $\square$

*Remark* 8.1. As seen in the proof, one has $\pi^*(\lambda_{6,2}(1)) = \widetilde{B} + A + \pi^*\lambda_{6,2}(c_1(L))$ for some line bundle $L \in Pic(J)$. As it was pointed out to me by Rapagnetta, using our construction one can easily show that there is a line bundle $A \in Pic(\widetilde{M}_6)$ such that $2A = \widetilde{\Sigma}$. Indeed, as shown in [OG2], we have

$$H^2(\widetilde{M}_6, \mathbb{Q}) = \widetilde{\mu}(H^2(J, \mathbb{Q})) \oplus \mathbb{Q} \cdot c_1(\widetilde{B}) \oplus \mathbb{Q} \cdot c_1(\widetilde{\Sigma}),$$

so that there are $\beta \in H^2(J, \mathbb{Q})$ and $p, q \in \mathbb{Q}$ such that

$$c_1(\pi^*\lambda_{6,2}(1)) = \widetilde{\mu}(\beta) + p\widetilde{B} + q\widetilde{\Sigma}.$$

By equation 7.3.5 in [OG2] one gets

$$0 = c_1(\pi^*\lambda_{6,2}(1)) \cdot \delta = p - 2q,$$

$$-1 = c_1(\pi^*\lambda_{6,2}(1)) \cdot \gamma = -2p + 2q.$$

In conclusion $q = 1/2$ and $p = 1$. Now, $c_1(\pi^*\lambda_{6,2}(1)) \in H^2(\widetilde{M}_6, \mathbb{Z})$, so that if $\widetilde{\Sigma}$ was a generator for $H^2(\widetilde{M}_6, \mathbb{Z})$, we would have $q \in \mathbb{Z}$, which is clearly not the case. Then, there must be a line bundle $A \in Pic(\widetilde{M}_6)$ such that $2c_1(A) = c_1(\widetilde{\Sigma})$, and we are done.

# 9 The Beauville form of $\widetilde{M}_6$

In this last section, we prove an analogue of Theorem 5.8 about the Beauville form of $\widetilde{M}_6$. Here is the result:

**Theorem 9.1.** *Let $v = (2, 0, -2) \in \widetilde{H}(J, \mathbb{Z})$. There is a morphism of Hodge structures*

$$f : v^\perp \longrightarrow H^2(\widetilde{M}_6, \mathbb{Z}),$$

*which is an isometry between $v^\perp$, as a sublattice of the Mukai lattice $\widetilde{H}(J, \mathbb{Z})$, and its image in $H^2(\widetilde{M}_6, \mathbb{Z})$, lattice with respect to the Beauville form $q$.*

*Proof.* As in Lemma 5.7, a Mukai vector $w$ is orthogonal to $v$ if and only if $w = (r, c, r)$ for $r \in \mathbb{Z}$ and $c \in H^2(J, \mathbb{Z})$, so that $v^\perp \simeq H^2(J, \mathbb{Z}) \oplus \mathbb{Z}$. Let

$$f : v^\perp \longrightarrow H^2(\widetilde{M}_6, \mathbb{Z}), \qquad f((r, c, r)) := \widetilde{\mu}(c) + rc_1(\widetilde{B}) + rc_1(A).$$

The morphism $f$ is an injective morphism of Hodge structures. By point 5 of Theorem 2.2 and definition of the Mukai pairing, it is easy to see that $f$ is an isometry on its image. $\square$

# 10 Appendix: Construction of flat families

In this Appendix we resume a general construction of flat families we used in several occasions. Let $S$ be an algebraic surface and $T$ a proper scheme. Let $p_S$ and $p_T$ be the two obvious projections of $T \times S$. Let $\mathscr{V}$ and $\mathscr{W}$ be two $T-$flat coherent sheaves on $T \times S$, and suppose $p_{T*}\mathscr{H}om(\mathscr{V}, \mathscr{W})$ to be a vector bundle on $T$. Let $Y := \mathbb{P}(p_{T*}\mathscr{H}om(\mathscr{V}, \mathscr{W}))$ and $p : Y \longrightarrow T$ the projection morphism. Let $\mathscr{T}$ be the tautological line bundle on $Y$. As shown in [H], Chapter II, Prop. 7.11, there is a canonical morphism $f : \mathscr{T} \longrightarrow p^* p_{T*}\mathscr{H}om(\mathscr{V}, \mathscr{W})$ which is injective. Finally, let $q_Y : Y \times S \longrightarrow Y$ and $q_S : Y \times S \longrightarrow S$ be the two projections. We have the following commutative diagram:

$$\begin{tikzcd} Y \arrow[d, "p"'] & Y \times S \arrow[l, "q_Y"'] \arrow[r, "q_S"] \arrow[d, "p \times id_S"] & S \arrow[d, equal] \\ T & T \times S \arrow[l, "p_T"] \arrow[r, "p_S"'] & S \end{tikzcd}$$

and the following equality holds:

$$p^* p_{T*}\mathscr{H}om(\mathscr{V}, \mathscr{W}) = q_{Y*}\mathscr{H}om((p \times id_S)^*\mathscr{V}, (p \times id_S)^*\mathscr{W}).$$

By the projection formula $f$ defines then a global section

$$\sigma \in H^0(Y \times S, \mathscr{H}om((p \times id_S)^*\mathscr{V} \otimes q_Y^*\mathscr{T}, (p \times id_S)^*\mathscr{W})),$$

corresponding to a morphism

$$\widetilde{f} : (p \times id_S)^*\mathscr{V} \otimes q_Y^*\mathscr{T} \longrightarrow (p \times id_S)^*\mathscr{W}.$$

This construction allows us to produce flat families, as shown in section 2.2 and 6.2.


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